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LETTER TO THE EDITOR

Solutions of the inhomogeneous acoustic–gravity wave equation

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Abstract. The response of a stratified compressible fluid (an 'atmosphere') to an oscillatory point source has in the past been discussed in an astrophysical context, with a view to increasing understanding of the complex problem of wave generation by turbulence. Much use has been made in such analyses of a technique developed by Lighthill in 1960; namely a method of determining the asymptotic functional form of the solution of the governing differential equation. In this analysis the wave equation for acoustic–gravity waves is solved exactly by three-fold Fourier integral techniques in a manner similar to that of Rao. Use is made of Lighthill's radiation condition to pick the correct physical solution.

The governing equation for the normalised pressure perturbation $\psi = p\rho_0^{-1/2}$ for motion induced by an as yet unspecified source in an isothermal stratified compressible unbounded fluid is (Kato 1966)

$$\left(c^2\nabla^2\frac{\partial^2}{\partial t^2}-\frac{\partial^4}{\partial t^4}-\omega_1^2\frac{\partial^2}{\partial t^2}+N^2c^2\nabla_1^2\right)\psi = S(x, y, z, t). \tag{1}$$

For a source term $f = (f_x, f_y, f_z)$ in the equation of motion the form of $S(x, y, z, t)$ is (Kato 1966)

$$S(x, y, z, t) = -\rho_0^{-1/2}c^2\left[\frac{\partial^2}{\partial t^2}\left(\frac{N^2}{g}+\frac{\partial}{\partial z}\right)f_z + \left(\frac{\partial^2}{\partial t^2}+N^2\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)\right] \tag{2}$$

where $\rho_0(z)$ is the equilibrium density distribution, c is the velocity of sound, $\boldsymbol{g} = (0, 0, -g)$ is the acceleration due to gravity, $\omega_1 = \gamma g/2c$ is the atmospheric acoustic cut-off frequency (γ is the ratio of specific heats), $N = (\gamma-1)^{1/2}g/c$ is the Brunt–Vaisala frequency, and $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Physically, the use of the variable $p\rho_0^{-1/2}$ rather than p is to factor out the amplitude variations required by conservation of energy of the wave as it moves into regions of different density. Mathematically, it corresponds to the removal of a first-order space-derivative term in equation (1) (i.e. the poles in the integrand of equation (3) below are made to lie on the real axis).

For the term f vanishing outside a restricted region, representing a localised 'forcing function', the problem can be solved in terms of a three-fold Fourier integral (Lighthill 1960):

$$\psi = \exp(\mathrm{i}\omega t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathcal{S}(l, m, n)\exp[\mathrm{i}(lx+my+nz)]}{G(l, m, n)}\,\mathrm{d}l\,\mathrm{d}m\,\mathrm{d}n \tag{3}$$

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and

$$S(x, y, z, t) = \exp(\mathrm{i}\omega t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathcal{S}(l, m, n) \exp[\mathrm{i}(lx + my + nz)]}{G(l, m, n)} \, \mathrm{d}l \, \mathrm{d}m \, \mathrm{d}n \tag{4}$$

where we have ascribed time-harmonic behaviour to the source, and

$$G(l, m, n) = \omega^2 c^2 \left[n^2 + (l^2 + m^2)\left(1 - \frac{N^2}{\omega^2}\right) + \frac{\omega_1^2 - \omega^2}{c^2} \right]. \tag{5}$$

We investigate solutions of equations (1) and (2) for the case

$$f_i = A\delta(x)\delta(y)\delta(z)\delta_{i3} \qquad i = 1, 2, 3; A \text{ constant}$$

i.e. a point disturbance oscillating vertically.

The Fourier transform of

$$\rho_0^{-1/2} c^2 \omega^2 A \left(\frac{N^2}{g} + \frac{\partial}{\partial z} \right) \delta(x)\delta(y)\delta(z)$$

is

$$\frac{\rho_0^{-1/2} c^2 \omega^2 A}{8\pi^3 g} (N^2 + \mathrm{i}ng).$$

Therefore we may write

$$\psi = \frac{\rho_0^{-1/2} A}{8\pi^3 g} \exp(\mathrm{i}\omega t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[\mathrm{i}(lx + my)] \, \mathrm{d}l \, \mathrm{d}m \int_{-\infty}^{\infty} \frac{(N^2 + \mathrm{i}ng)}{G'} \exp(\mathrm{i}nz) \, \mathrm{d}n \tag{6}$$

where

$$G' = n^2 - n_0^2 \tag{7}$$

and

$$n_0 = \left[(l^2 + m^2)\left(\frac{N^2}{\omega^2} - 1 \right) + \frac{\omega^2 - \omega_1^2}{c^2} \right]^{1/2}. \tag{8}$$

Lighthill's radiation condition (Lighthill 1960) consists of replacing ω by $\omega - \mathrm{i}\epsilon$, $\epsilon > 0$, the source strength being taken as $\exp[\mathrm{i}(\omega - \mathrm{i}\epsilon)t] S(x, y, z)$. This provides a unique solution to the original problem in the limit $\epsilon \to 0$.

The real simple poles occur at

$$n = \pm n_0 + \mathrm{i}\epsilon \frac{\partial G'/\partial \omega}{\partial G'/\partial n}$$

and the displacement is into the upper or lower half-plane according to whether

$$\frac{\partial G'/\partial \omega}{\partial G'/\partial n} \gtrless 0.$$

If we take $\omega > 0$, then n_0 is displaced into the lower half-plane ($-n_0$ into the upper half-plane) provided $\omega^4 > N^2 c^2 (l^2 + m^2)$; and n_0 is displaced into the upper half-plane ($-n_0$ into the lower half-plane) provided $\omega^4 < N^2 c^2 (l^2 + m^2)$.

For a semicircular contour taken in the upper half-plane only

$$n = -n_0 + \mathrm{i}\epsilon \frac{\partial G'/\partial \omega}{\partial G'/\partial n}$$

gives a contribution for $\omega^4 > N^2c^2(l^2+m^2)$. The residue at this pole is to be calculated as $\epsilon \to 0$. The case $\omega^4 < N^2c^2(l^2+m^2)$ can obviously be considered in a similar manner.

Now

$$\int_{-\infty}^{\infty} \frac{(N^2+ing)\exp(inz)}{n^2-n_0^2}\,dn$$

$$= -i\pi\left\{N^2\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{-1/2} - ig\right\}\exp\left\{-i\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{1/2} z\right\} \qquad (9)$$

where

$$s^2 = l^2+m^2, \qquad \tilde{\omega}^2 = -\omega^2+\omega_1^2.$$

Let

$$l = s\cos\phi, \quad m = s\sin\phi; \qquad x = r\cos\theta, \quad y = r\sin\theta$$

hence the solution is

$$\psi = -\frac{i\rho_0^{-1/2}AN^2\exp(i\omega t)}{8\pi^2 g} I_1 - \frac{\rho_0^{-1/2}A\exp(i\omega t)}{8\pi^2} I_2 \qquad (10)$$

where

$$I_1 = \int_0^\infty \int_0^{2\pi} s\,ds\,d\phi\,\exp\left\{-iz\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{1/2}\right\}$$

$$\times\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{-1/2} \exp[irs\cos(\phi-\theta)] \qquad (11)$$

and

$$I_2 = \int_0^\infty \int_0^{2\pi} s\,ds\,d\phi\,\exp\left\{-iz\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{1/2}\right\}\exp[irs\cos(\phi-\theta)]. \qquad (12)$$

Therefore

$$I_1 = \int_0^\infty \exp\left\{-iz\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{1/2}\right\}\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{1/2}\Big\}J_0(rs)s\,ds \qquad (13)$$

$$I_2 = \int_0^\infty \exp\left\{-iz\left[s^2\left(\frac{N^2}{\omega^2}-1\right)-\frac{\tilde{\omega}^2}{c^2}\right]^{1/2}\right\}J_0(rs)s\,ds. \qquad (14)$$

Note that equations (13) and (14) also follow from operating with a Fourier–Hankel transform on equations (1) and (2), and hence it is possible to formulate axisymmetric problems in terms of Bessel functions, in a manner analogous to that of Lighthill (1960).

These integrals can be evaluated using results given in Erdélyi *et al* (1954). Since we are only interested in the qualitative behaviour of $\psi(\omega; r, z)$ we write down the general spatial behaviour only; hence

$$I_1 \sim (z^2-\Gamma^2r^2)^{-1/2}\exp\left(-\frac{\tilde{\omega}}{c}(z^2-\Gamma^2r^2)^{1/2}\right) \qquad (15a)$$

$$= -i(\Gamma^2r^2-z^2)^{-1/2}\exp\left(-i\frac{\tilde{\omega}}{c}(\Gamma^2r^2-z^2)^{1/2}\right) \qquad (15b)$$

$$I_2 \sim r^{1/2}(z^2 - \Gamma^2 r^2)^{-3/4} K_{3/2}\left(\frac{\tilde{\omega}}{c}(z^2 - \Gamma^2 r^2)^{1/2}\right) \qquad (16a)$$

$$= (-1)^{3/4} r^{1/2}(\Gamma^2 r^2 - z^2)^{-3/4} K_{3/2}\left(\mathrm{i}\frac{\tilde{\omega}}{c}(\Gamma^2 r^2 - z^2)^{1/2}\right) \qquad (16b)$$

where $\Gamma^2 = \omega^2(N^2 - \omega^2)^{-1}$, and $K_\nu(x)$ is a modified Bessel function of the second kind of order ν, argument x.

Results (15*a*) and (16*a*) are appropriate if $\Gamma^2 < 0$, while results (15*a*) and (16*a*) are appropriate for $z > \Gamma r$ and $\Gamma^2 > 0$. Results (15*b*) and (16*b*) are appropriate for $z < \Gamma r$ and $\Gamma^2 > 0$. We are now in a position to distinguish between three regimes.

(*a*) *Buoyancy dominated motions* (*i.e. compressibility-modified gravity waves*): $\omega^2 < N^2$

In this regime $\tilde{\omega}$ is real and $\Gamma^2 > 0$, so it is seen that disturbances decay for $z > \Gamma r$ and are oscillatory in nature for $z < \Gamma r$. The physical reason for this is best understood by noting that internal gravity waves have not only a characteristic frequency, but a characteristic direction (gravity). Their propagation is thus markedly anisotropic, and the corresponding wavenumber surface is asymptotically a cone with half-angle $\theta = \sin^{-1}(\omega/N)$ about the vertical—the complete surface being a hyperbola of revolution (Moore and Spiegel 1964). Thus gravity waves may only propagate when $\omega < N$ and they cannot propagate vertically ($s = 0$) since in the absence of any horizontal variation there would be no buoyancy force. Thus the quantity Γ represents the tangent of the angle between the wavevector and the horizontal space axis.

(*b*) *Compressibility dominated motions* (*i.e. buoyancy-modified sound waves*): $\omega^2 > \omega_1^2$

In this regime $\tilde{\omega}$ is imaginary and $\Gamma^2 < 0$, so the disturbances are oscillatory for all real z and r. This merely reflects the fact that, even under gravity, acoustic waves propagate nearly isotropically.

(*c*) *Trapped modes* (*horizontally propagating*): $N^2 < \omega^2 < \omega_1^2$

Here $\tilde{\omega}$ is real and $\Gamma^2 < 0$. Since, for real x, $K_\nu(x)$ is a monotonically decreasing function (whereas for imaginary x it is oscillatory) it is seen that disturbances in this regime decay in z for all real z and r. These are modes 'trapped' in the evanescent region in ω–s space between the acoustic and buoyancy cut-off frequencies. Note that the oscillatory terms in equations (15) and (16) represent waves with surfaces of constant phase, $z^2 - \Gamma^2 r^2 =$ constant, which are hyperbolas asymptotic to the boundary of the region $z < \Gamma r$ (if $\Gamma^2 > 0$). There is an exponential decay of disturbance energy in the region $z > \Gamma r$ for such buoyancy-dominated motions. A similar decay occurs within regime (*c*).

In conclusion therefore we have obtained exact closed-form solutions to the acoustic–gravity wave equation with a simple source term. In so doing we have been able to consider the solution of the problem in some mathematical detail. This extends and complements the asymptotic treatment of Moore and Spiegel (1964), and the numerical approach of Kato (1966). In addition we note that since the effects of compressibility were neglected by Rao (1973), its inclusion will introduce more modes in the manner of those discussed here.

References

Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1954 *Tables of Integral Transforms* vol. 2 (New York: McGraw-Hill) p 31
Kato S 1966 *Astrophys. J.* **143** 893
Lighthill M J 1960 *Phil. Trans. R. Soc.* A **252** 397
Moore D W and Spiegel E A 1964 *Astrophys. J.* **139** 48
Ramachandra Rao A 1973 *J. Fluid Mech.* **58** 161